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Stable Stationary Solutions Induced by Spatial Inhomogeneity Via Γ-Convergence

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— Dedicated to the memory of R. Mañé

Abstract. Using among other tools an approach based on the variational concept of Γ-convergence, we manage to prove existence as well as stability and exhibit the geometric structure of a family of stationary solutions of a semilinear diffusion equation. The existence of these stable stationary solutions is solely due to suitable oscillations of the functions characterizing the spatial inhomogeneities involved in the problem. In particular,these oscillations depend on the signed curvature of a level curve of the square root of the product of these functions.

Keywords: Stable equilibria, Spatial Inhomogeneity, Gamma-convergence, Reaction-diffusion equations.

1. Introduction

This paper will focus on existence, stability and geometric structure of some stationary solutions of the following problem:

$$\begin{aligned} &\frac{\partial v}{\partial t} = \varepsilon^2 \operatorname{div}[k_1(X)\nabla v] + k_2(X)f(v)\,,\ (t,X) \in \mathbb{R}^+ \times \Omega \\ &v(0,X) = v_0(X) \in H^1(\Omega) \\ &\nabla v(t,X)\cdot \widehat{n}_1(X) = 0 \quad \text{for} \quad (t,X) \in \mathbb{R}^+ \times \partial\Omega \end{aligned} \tag{P$_1$}$$

where $X = (x_1, x_2) \in \Omega \subset \mathbb{R}^2$; Ω bounded open region with C^3 boundary; $\widehat{n}_1$: outward pointing unit normal vector to $\partial\Omega$, ε : small positive parameter; k_1 and k_2 are strictly positive functions in $C^2(\bar{\Omega}, \mathbb{R}^+)$ and f is a real C^1-function which satisfies:

(f_0) $| f(v) | \leq a + b| v |^\sigma$, for some constants a, b and σ, $1 \leq \sigma < \infty$.

(f_1) f has three consecutive zeros α, θ and β, $-\infty < \alpha < \theta < \beta <$

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$+\infty$ with $f'(\alpha) < 0$ and $f'(\beta) < 0$.

(f$_2$) $F(\alpha) = F(\beta)$, where $F(v) = \int_\theta^v f$.

The question of how the diffusivity function k_1 (terminology borrowed from the reaction-diffusion context) can give rise to stable nonconstant stationary solutions of (P_1), has been subject of research for some time. All the works though are restricted to the onedimensional spatial variable case. See [**HR**], [**FH**] and [**N1**]. Herein we manage to treat the two-dimensional spatial variable case using a technique known as Γ-convergence. Actually the aim of this paper is two-fold: first it provides a procedure for obtaining existence as well as the geometric qualitative structure and stability of a family of stationary solutions of (P_1). Second it describes the mechanism of interaction between the spatial inhomogeneities k_1 and k_2 and the reaction term f which is ultimately responsible for the existence and stability of the equilibrium solutions obtained. When $k_1(X) =$ const., $k_2(X) =$ const. and Ω is convex then it has long been known that (P_1)possesses no nonconstant stable stationary solutions if ε is small enough. However when Ω has "necks" or exhibts a "dumbbell-shape", such solutions do exist. See [**M**], [**HV**] and [**KS**]. Herein the functions k_1 and k_2 play the same role the geometry of $\partial\Omega$ does in the references above. In [**N1**] this question was addressed for the case in which $\Omega = B_1(0)$: the unit ball with center at the origin, k_1 and k_2 are radially symmetric functions. When $k_2 \equiv$ const. and ε small , a diffusivity function $k_1(\|X\|;\varepsilon)$ was found so that (P_1)possesses an uniformly asymptotically stable radially symmetric stationary solution which develops boundary or spike layer formation accordingly to whether $F(\alpha) > F(\beta)$ or $F(\alpha) < F(\beta)$. Note that k_1 depended on ε. Herein due to the Γ-convergence approach, once a diffusivity function $k_1(X)$, $X \in \Omega$, satisfies hypothesis (H_1) below, the existence of a family of stable stationary solution $\{v_\varepsilon\}$ is guaranteed for all ε, $0 < \varepsilon < \varepsilon_0$, for some $\varepsilon_0 > 0$.

Next we describe our results as a theorem.

Theorem 1.1. *Let:*

1.1.i) $k(X) \triangleq (k_1(X)k_2(X))^{1/2}$, *$k_1$ and k_2 as in (P_1), ν_m some value in* *the range of k and $\gamma(s)$, $0 \leq s \leq L$, be an arc-length parametrized C^2 simple closed curve such that*

$$\gamma \subseteq k^{-1}(\nu_m) = \{X \in \Omega : k(X) = \nu_m\}.$$

1.1.ii) $\kappa(s)$, $0 \leq s \leq L$, *be the signed curvature of* γ

1.1.iii) $\Omega = \Omega_i \cup \Omega_o \cup \gamma$, *where* Ω_i *stands for the open region in* Ω *enclosed by* γ *and* $\Omega_o = \Omega \setminus (\Omega_i \cup \gamma)$

1.1.iv) $\widehat{n}_2[\gamma(s)]$, $0 \leq s \leq L$, *be the inward-pointing normal vector to* γ *and consider the following change of coordinates:* $\Sigma : Q_\delta \to N_\delta$, *given by*

$$X = (x_1, x_2) = \Sigma(s,t) = \gamma(s) + t\widehat{n}_2[\gamma(s)]$$

$0 \leq s \leq L$, $-\delta < t < \delta$, *where* N_δ *is a small tubular neighborhood of* γ, $Q_\delta = \{(s,t) \in \mathbb{R}^2 : 0 < s < L, -\delta < t < \delta\}$ *with* δ *small enough so that* Σ *is a diffeomorphism. Set* $\tilde{k}(s,t) = k(\Sigma(s,t))$ *and suppose that:*

$$\begin{aligned} \frac{\partial \tilde{k}(s,0)}{\partial t} &= \nu_m \,\kappa(s), \;\; 0 \leq s \leq L \\ \frac{\partial^2 \tilde{k}(s,0)}{\partial t^2} &> 2\nu_m \,\kappa^2(s), \;\; 0 \leq s \leq L \end{aligned} \tag{H$_1$}$$

Then there is a family of stationary solutions $\{v_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ *of* (P_1)*such that:*

1.1) v_ε *is stable, for* $0 < \varepsilon \leq \varepsilon_0$

1.2) $\alpha < v_\varepsilon(X) < \beta$, $X \in \overline{\Omega}$

1.3) *For any compact set* $K \subset \Omega_i (K \subset \Omega_0)$ *it holds that* $v_\varepsilon \to \beta$ $(v_\varepsilon \to \alpha)$, *uniformly on* K, *as* $\varepsilon \to 0$. *In particular the level curve* $\gamma_\varepsilon = \{X \in \Omega : v_\varepsilon(X) = \theta\}$ *satisfies* $\gamma_\varepsilon \to \gamma$, *uniformly, as* $\varepsilon \to 0$.

See Figure 1 for an illustration of how $\tilde{k}$ should behave around γ, as κ changes sign.

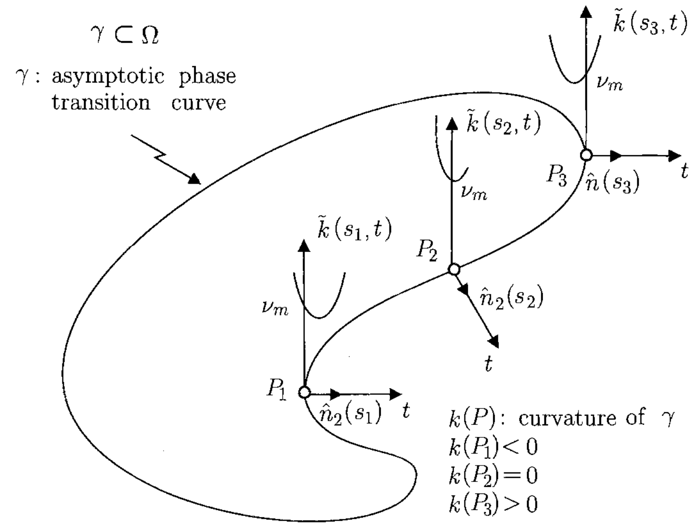

Figure 1

Remark 1.1. Note that $\tilde{k}(s,0) = \nu_m$, but γ is not a curve of local minima for $\tilde{k}(s,t)$. Instead γ will be a curve of local minima for the function $\Lambda(s,t) = (1 - t\kappa(s))\tilde{k}(s,t)$. An example of such a function $\tilde{k}$ can be easily constructed. Take, for instance, $\gamma \subset \Omega \subset \mathbb{R}^2$ with $\gamma = \{X \in \Omega : ||X|| = 1\}$. Then the function $\tilde{k}(s,t) = 1 + t + 2t^2$, $t \in (-\delta, \delta)$, δ: small, $s \in [0, 2\pi)$, satisfies (H_1) with $\nu_m = 1$.

Remark 1.2. When $\Omega = (a,b)$, then it has long been known that the roles played by k_1 and k_2 are interchangeable. Actually this can be accomplished by performing the following change of spatial variable,

$$y = \tau(x) = \int_0^x \left(\frac{1}{k_1^2(t)}\right) dt.$$

It is clear from Theorem 1.1 that this feature is also carried on to the two-dimensional case. As it will be seen later on, this is possible by virtue of the special form of a limiting problem, in the sense of Γ-convergence, that will be crucial to our analysis.

Remark 1.3. It should also be emphazised that since all works up to now deal only with the onedimensional case they provide no clue of the role played by the limiting (when $\varepsilon = 0$) transition layer curve γ. Regarding this matter it is required through (H_1) that as a point X moves along γ counterclockwise the directional derivative $\nabla k(X) \cdot \widehat{n}_2(X)$ changes sign whenever the curvature of γ does so while the concavity of $k(\Sigma(s,t))$, seen as a function of t alone, is always positive and becomes greater as the absolute value of κ does so.

Remark 1.4. Suppose, for the sake of argument, that $k_2 \equiv 1$ and that (P_1)governs the diffusion of a substance in a medium Ω whose diffusivity function is given by $k_1(X)$, $X \in \Omega$. So in order to exist a stable stationary solution , induced by k_1 , (H_1) reflects the physically reasonable requirement that k_1 should properly increase (decrease) across the transition layer whenever the corresponding stationary solution decreases (increases) along the same direction. It also shows how this variation depends on the curvature of the limiting interface ($\varepsilon = 0$) and actually it is necessary to offset the tendency of the diffusing substance to spread homogeneously in space and eventually setting down in a constant concentration over Ω.

Next we make precise what we mean by stability of a stationary solution of (P_1). For simplicity, let $Y = H^1(\Omega)$ and denote by $\|\cdot\|$ its usual norm.

Definition 1.1. Let $\tilde{v}$ be a solution of (P_2)in X. We say that $\tilde{v}$ is a stable stationary solution of (P_1)if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $v_0 \in X$, $\|v_0 - \tilde{v}\| < \delta$ then the solution $v(X,t)$ of (P_1)so that $v(X,0) = v_0$ exists for all $t \geq 0$ and satisfies $\|v(\cdot\,, X) - \tilde{v}(X)\| < \varepsilon$.

If in addition to the property above it also holds that $\lim_{t\to\infty} v(X,t) = \tilde{v}(X)$ in Y, then $\tilde{v}$ is said to be strongly stable.

A few words about our approach: when looking for stationary solutions of (P_1)it suffices to look for minimisers of a certain family of functionals E_ε, defined in $L^1(\Omega)$, through a penalization. By taking the Γ-limit, as $\varepsilon \to 0$, we end up with a more tractable geometrical problem of minimising the Γ-limiting functional E_0 in $BV(\Omega)$. It turns out that close, in the sense of the topology of $L^1(\Omega)$, to any isolated L^1-local minimiser of E_0 there corresponds a minimiser of E_ε, which by its turn is a stable stationary solution of (P_1).

2. Preliminaries on $BV(\Omega)$ and Γ-convergence

We need some definitions and notations about functions of bounded variation in $\mathbb{R}^n$. For further background the reader is referred to [Z], [EG] and [Giu], for instance. Let $v \in L^1(\Omega)$, with Ω a bounded Lipschitz domain in $\mathbb{R}^n$.

Definition 2.1. v is a function of bounded variation in Ω if its partial derivatives in the sense of distributions are measures with finite total variation in Ω.

In the sense of distribution Dv is a vector valued Radon measure with finite total variation in Ω given by

$$\|Dv\|(\Omega) = \sup_{\substack{\sigma \in C_0^1(\Omega,\mathbb{R}^n) \\ |\sigma| \leq 1}} \int_\Omega v(X) \operatorname{div} \sigma(X) dX .$$

The total variation $\|Dv\|$ is a measure itself. A Borel set $B \subset \mathbb{R}^n$ has finite perimeter in the open set Ω if

$$\mathrm{Per}_\Omega(B) = \|D\mathcal{X}_B\|(\Omega) < \infty ,$$

where $\mathcal{X}_B$ is the characteristic function of B.

If $k \in C^1(\Omega, \mathbb{R}^+)$ and $v \in BV(\Omega)$, then the integral of $k(X)$ with respect to the measure $\|Dv\|$ is defined as (see [**Fe**], for instance)

$$\int_\Omega k(X) d\|Dv\|(X) = \sup_{\substack{\sigma \in C_0^1(\Omega,\mathbb{R}^n) \\ |\sigma(X)| \leq k(X)}} \int_\Omega v(X) \operatorname{div} \sigma(X) dX .$$

If $A \subset \Omega$ has smooth boundary ∂A then the divergence theorem implies

$$\int_\Omega k(X) d\|D\mathcal{X}_A\|(X) = \int_{\Omega \cap \partial A} k(X) d\mathcal{H}^{n-1}(X)$$

where $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure. This provides a geometrical idea of the kind of integral we will be dealing with.

Above references supply definitions of approximate continuity, approximate differentiability, bounded essential variation, etc., which will be used later on.

Definition 2.2. A family $\{E_\varepsilon\}_{\varepsilon>0}$ of real-extended functionals defined in $L^1(\Omega)$ is said to Γ-converge, as $\varepsilon \to 0$, to a functional E_0, at v and we write

$$\Gamma\left(L^1(\Omega)^-\right) - \lim_{\varepsilon\to 0} E_\varepsilon(v) = E_0(v)$$

if:

- For each $v \in L^1(\Omega)$ and for any sequence $\{v_\varepsilon\}$ in $L^1(\Omega)$ such that $v_\varepsilon \to v$ in $L^1(\Omega)$, as $\varepsilon \to 0$, implies $E_0(v) \leq \liminf_{\varepsilon\to 0} E_\varepsilon(v_\varepsilon)$
- For each $v \in L^1(\Omega)$ there is a sequence $\{w_\varepsilon\}$ in $L^1(\Omega)$ such that $w_\varepsilon \to v$ in $L^1(\Omega)$, as $\varepsilon \to 0$, and $E_0(v) \geq \limsup_{\varepsilon\to 0} E_\varepsilon(w_\varepsilon)$

Definition 2.3. We shall call $v_0 \in L^1(\Omega)$ an L^1-local minimiser of E_0 if there is $\mu > 0$ such that

$$E_0(v_0) \leq E_0(v) \quad \text{whenever} \quad 0 < \|v - v_0\|_{L^1(\Omega)} < \mu\,.$$

Moreover if $E_0(v_0) < E_0(v)$ for $0 < \|v - v_0\|_{L^1(\Omega)} < \mu$, then v_0 is called an isolated L^1-local minimiser of E_0.

The following theorem is due in its abstract form to De Giorgi [**Gio**] and in its variational form can be found in [**KS**].

Theorem 2.1. *Suppose that a sequence of real-extended functionals $\{E_\varepsilon\}$, Γ-converges to a real-extended functional E_0 and also that the following hypotheses are satisfied:*

2.1.i) *Any sequence $\{v_\varepsilon\}_{\varepsilon>0}$ such that $E_\varepsilon(v_\varepsilon) \leq C < \infty$ for all $\varepsilon > 0$, is compact in $L^1(\Omega)$.*

2.1.ii) *There exists an isolated L^1-local minimiser v_0 of E_0.*

Then there exists an $\varepsilon_0 > 0$ and a family $\{v_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ such that:

- *v_ε is an L^1-local minimiser of E_ε.*
- *$\|v_\varepsilon - v_0\|_{L^1(\Omega)} \to 0\,,$ as $\varepsilon \to 0$.*

Forseeing a future application to our case, we define a family of functionals $E_\varepsilon : L^1(\Omega) \to \mathbb{R} \cup \{\infty\}$ by:

$$E_\varepsilon(v) = \begin{cases} \displaystyle\int_\Omega \left\{\frac{\varepsilon}{2} k_1(X)\,|\nabla v|^2 + \varepsilon^{-1} k_2(X) W(v)\right\} dX\,, & \text{if } v \in H^1(\Omega) \\ \infty\,, & \text{otherwise} \end{cases}$$

where W satisfies any growth condition that makes E_ε well defined. We will come to this point later in the beginning of Section 3. There this condition will be satisfied by virtue of (f_0).

The following theorem can be found in [**S**]. The presence of the function $k_1(X)$ adds no additional difficulties to the proof. See also [**OS**].

Theorem 2.2. *Suppose that in the above-defined family of functionals E_ε, the potential function $W(v)$ satisfies:*

2.2.i) $W \in C^2$, $W \geq 0$.

2.2.ii) W *has exactly two roots α and β, with $\alpha < \beta$.*

2.2.iii) $W'(\alpha) = W'(\beta) = 0$, $W''(\alpha) > 0$, $W''(\beta) > 0$.

Then $\Gamma\left(L^1(\Omega)^-\right)$ - $\lim_{\varepsilon \to 0} E_\varepsilon(v) = E_0(v)$, *where*

$$E_0(v) = \begin{cases} C_0 \int_\Omega (k_1(X)\, k_2(X))^{1/2}\, d\left\|D\mathcal{X}_{\{v=\beta\}}\right\|, & \text{if } v \in BV(\Omega), \\ v(X) \in \{\alpha, \beta\} \quad \text{a.e. in}\Omega\,, 0 < |\{X : v(X) = \beta\}| < |\,\Omega| & \\ \infty, & \text{otherwise} \end{cases}$$

where

$$C_0 = \int_\alpha^\beta W^{1/2}(s) ds\,.$$

In order to apply Theorem 2.1 we have to exibit an isolated L^1-local minimizer of E_0. This is the core of our analysis and the object of our next theorem.

Theorem 2.3. *Suppose that hypotheses 1.1.i) - 1.1.iv) and (H_1) of Theorem 1.1 hold. Then the function*

$$v_0(X) = \beta\, \mathcal{X}_{\Omega_o} + \alpha \mathcal{X}_{\Omega_i}$$

is an isolated L^1-local minimiser of E_0, given above.

The following lemmas whose proofs are straightfoward will be needed in the proof of Theorem 2.3.

Lemma 2.1. *Let $\Lambda(s,t) = \tilde{k}(s,t) J_\Sigma(s,t)$, $(t,s) \in Q_\delta$, where $\tilde{k}$ is as in (1.1.iv) and $J_\Sigma(s,t) = (1 - t\kappa(s))$ the Jacobian of Σ as in (1.1.iv). Then* *it holds that* $\Lambda(s,t) > \Lambda(s,0) = \nu_m$ *, for any* $(t,s) \in Q_\delta$*,* $t \neq 0$*. Moreover if* δ *small enough,* $J_\Sigma(s,t) > 0$ *in* Q_δ *.*

Lemma 2.2. *Let* γ, Σ *and* κ *be as in Theorem 2.3 and* $M_\Sigma(s,t)$ *stand for the Jacobian matrix of* Σ *. If* $w \in C^1(N_\delta, \mathbb{R})$ *and* $\tilde{w}(s,t) = w(\Sigma(s,t))$ *then*

$$\left|M_\Sigma^{-1}(s,t)\nabla_{s,t}\tilde{w}\right| = \left|\left(\frac{\partial\tilde{w}}{\partial s}\right)^2 (1-t\kappa(s))^{-2} + \left(\frac{\partial\tilde{w}}{\partial t}\right)^2\right|^{1/2},$$

where $(1-t\kappa(s)) = J_\Sigma(s,t) > 0$ *in* Q_δ *.*

Proof of Theorem 2.3: It suffices to prove that if $0 < \|v - v_0\|_{L^1(N_\delta)} < \mu$, for suitable μ, then $E_0(v_0) < E_0(v)$, that is,

$$\int_{N_\delta} k(X)d\|Dv\|(X) > \int_{N_\delta} k(X)d\|Dv_0\|(X),$$

where the coarea formula was used to obtain the above expression.

In order to compute $E_0(v_0)$, set

$$Z_\xi = \{X \in N_\delta : v_0(X) > \xi\}$$

and note that $\partial\{X : v_0(X) > \alpha\} \cap N_\delta = \gamma$ is C^2. Hence the coarea formula yields:

$$\begin{aligned}
\int_{N_\delta} k(X)d\|Dv_0\|(X) &= \int_{-\infty}^{\infty}\left(\int_{N_\delta\cap\partial Z_\xi} k(X)d\mathcal{H}^1(X)\right)d\xi \\
&= (\beta-\alpha)\int_\gamma k(X)d\mathcal{H}^1(X) \\
&= (\beta-\alpha)\int_{\{(s,0),0\le s\le L\}} \Lambda(s,t)d\mathcal{H}^1(s,t) \\
&= (\beta-\alpha)\nu_m L = E_0(v_0).
\end{aligned}$$

Let " $\sim$ " denote, as before, a function in the new coordinates (s,t). Since $v \in BV(N_\delta)$ it follows that $\tilde{v} \in BV(Q_\delta)$. For a fixed $t \in (-\delta,\delta)$ define

$$\ell_t = \{(s,t),\ 0 < s < L\}.$$

Let L^1 stand for the one dimensional Lebesgue measure.

Hence the trace of $\tilde{v}(\cdot,t)$ is well defined on ℓ_t, for a.e. $t \in (-\delta,\delta)$, see [**Z**], for instance. In what follows the equality between two functions, along each ℓ_t, should be understood in the sense of equality between the traces of the two functions along ℓ_t.

Suppose now that:

i-) $\tilde{v} = \tilde{v}_0$ along $\ell_{\tilde{t}} \cup \ell_{-\tilde{t}}$, for some $\tilde{t} \in (\delta/2, \delta)$.

Consider a sequence $\{v_j\}_{j=1}^{\infty}$ in $C^{\infty}(N_\delta) \cap BV(N_\delta)$ such that $v_j \to v$ in $L^1(N_\delta)$ and

$$\int_{N_\delta} k(X) d\|Dv\|(X) = \lim_{j\to\infty} \int_{N_\delta} k(X)|\nabla v_j| dX .$$

For the case $k \equiv$ const. this can be found in [**Giu**], for instance, and in [**N2**] for the case above. See [**Fe**], for a more general case. Next take a subsequence of $\{v_j\}_{j=1}^{\infty}$, still denoted by $\{v_j\}_{j=1}^{\infty}$, such that $v_j \to v$, a. e. in N_δ.

Using Lemmas 2.1 and 2.2 and Fatou's lemma we obtain:

$$\begin{aligned}
\int_{N_\delta} k(X) d\|Dv\|(X) &= \lim_{j\to\infty} \int_{N_\delta} k(X)|\nabla v_j| dX \\
&= \lim_{j\to\infty} \iint\limits_{Q_\delta} \tilde{k}(s,t) J_\Sigma(s,t) \left| M_\Sigma^{-1}(s,t) \nabla_{(s,t)} \tilde{v}_j \right| dt\, ds \\
&\geq \liminf_{j\to\infty} \int_0^L \int_{-\delta/2}^{\delta/2} \Lambda(s,t) \left| \frac{\partial \tilde{v}_j}{\partial t} \right| dt\, ds \\
&\geq \nu_m \liminf_{j\to\infty} \int_0^L \int_{-\delta/2}^{\delta/2} \left| \frac{\partial \tilde{v}_j(s,t)}{\partial t} \right| dt\, ds \\
&\geq \nu_m \liminf_{j\to\infty} \int_0^L \mathrm{ess}V_{-\delta/2}^{\delta/2} \left[\tilde{v}_j(s,\cdot) \right] ds \\
&\geq \nu_m \int_0^L \mathrm{ess}V_{-\delta/2}^{\delta/2} \left[\tilde{v}(s,\cdot) \right] ds \\
&\geq \nu_m (\beta - \alpha) L = E_0(v_0) .
\end{aligned}$$

Here $\mathrm{ess}V_{-\delta/2}^{\delta/2}[\tilde{v}(s,\cdot)]$ stands for the total essential variation of $\tilde{v}(s,\cdot)$ on $(-\delta/2, \delta/2)$ and from the theory of BV functions $\mathrm{ess}V_{-\delta/2}^{\delta/2}[\tilde{v}(s,\cdot)]$ is Lebesgue integrable on $[0,L]$ for a.e. $t \in (-\delta/2, \delta/2)$ and

$$\liminf_{j\to\infty} \mathrm{ess}V_{-\delta/2}^{\delta/2} \left[\tilde{v}_j(s,\cdot) \right] \geq \mathrm{ess}V_{-\delta/2}^{\delta/2} \left[\tilde{v}(s,\cdot) \right]$$

for a.e. $s \in [0, L]$.

We claim that $E_0(v) > E_0(v_0)$ for if it were not the case then the above chain of inequalities along with the coarea formula would yield:

$$\begin{aligned}(\beta-\alpha)\nu_m L &= \int_{N_\delta} k(X)d\|Dv\|(X)\\ &= \int_{-\infty}^{\infty}\left(\int_{N_\delta\cap\partial\{v>\xi\}} k(X)d\|Dv\|(X)\right)d\xi\\ &= \int_{-\infty}^{\infty}\left(\int_{N_\delta\cap\partial^*\{v>\xi\}} k(X)d\mathcal{H}^1(X)\right)d\xi \qquad (2.1)\\ &= \int_{-\infty}^{\infty}\left(\int_{Q_\delta\cap\partial^*\{\tilde v>\xi\}} \Lambda(s,t)d\mathcal{H}^1(s,t)\right)d\xi\\ &= (\beta-\alpha)\int_{Q_\delta\cap\partial^*\{v=\beta\}\cap\partial^*\{v=\alpha\}} \Lambda(s,t)d\mathcal{H}^1(s,t)\end{aligned}$$

where the change of variables Σ was used. Note also that since $\partial\{v>\xi\}$ has locally finite perimeter in N_δ, then

$$\|\partial\{v>\xi\}\| = \mathcal{H}^1 \llcorner \partial^*\{v>\xi\},$$

where $\partial^* E$ stands for the reduced boundary of E (see [**EG**] for instance). Setting for convenience $S = \partial^*\{v=\alpha\}\cap\partial^*\{v=\beta\}$ it follows that

$$\int_{S\cap Q_\delta} \Lambda(s,t)d\mathcal{H}^1(s,t) = \nu_m L\,. \qquad (2.2)$$

Hypothesis i-) above means that

$$\beta = \tilde v(s,\tilde t) = \frac{1}{2}\left(\mu_{\tilde v}(s,\tilde t) + \lambda_{\tilde v}(s,\tilde t)\right)$$

$\mathcal{H}^1$-a.e. along $\ell_{\tilde t}$, where $\mu_{\tilde v}$ and $\lambda_{\tilde v}$ are the upper and lower approximate limits of $\tilde v$. See [**Z**], p.258, for instance.

But $\tilde v$ takes on only the values α and β on Q_δ and therefore

$$\beta = \mu_{\tilde v} = \lambda_{\tilde v}\,, \quad \mathcal{H}^1\text{- a.e. along } \ell_{\tilde t}\,.$$

Hence, $\tilde v = \mu_{\tilde v} = \lambda_{\tilde v}$, $\mathcal{H}^1$-a.e. along $\ell_{\tilde t}$, i.e., $\tilde v$ is approximate continuous $\mathcal{H}^1$-a.e. along $\ell_{\tilde t}$ and as such there follows the existence of a set $B\subset Q_\delta$, $|B|>0$, such that $\ell_{\tilde t}\subset B$ and $\tilde v(s,t)=\beta$ for any $(s,t)\in B$.

The very same argument applied to $\ell_{-\tilde t}$, implies the existence of a set A, $|A|>0$, such that $\ell_{-\tilde t}\subset A$ and $\tilde v(s,t)=\alpha$ for any $(s,t)\in A$.

Thus it follows rather easily from previous considerations that there are points $P_1 = (0, t_1)$, $P_2 = (L, t_2)$ where $-\tilde{t} \le t_1 \le t_2 \le \tilde{t}$, such that $\{P_1, P_2\} \subset C\ell_{Q_\delta} S$, where $C\ell_{Q_\delta} S$ stands for the closure of S in Q_δ.

But S has locally finite perimeter in Q_δ and therefore S is 1-rectifiable, i.e.,

$$S \subset \bigcup_{i=0}^{\infty} S_i,$$

where $\mathcal{H}^1(S_0) = 0$ and each S_i, $i = 1, 2, \ldots$ is an 1-dimensional embedded C^1 submanifold of $\mathbb{R}^2$. See [**Z**], for instance. This along with the fact that $P_i \in C\ell_{Q_\delta} S$, $i = 1, 2$, implies

$$\int_{S \cap Q_\delta} d\mathcal{H}^1(s,t) \ge L\,. \tag{2.3}$$

Recall that L is the total length of γ, the asymptotic interface curve. Inequality (2.2) along with equality (2.3) and Lemma 2.1 implies that $S = \{(s, 0),\ 0 < s < L\}$, i.e., $\tilde{v} = \tilde{v}_0$ a.e. in Q_δ.

This is a contradiction for we required that

$$\|\tilde{v} - \tilde{v}_0\|_{L^1(Q_\delta)} > 0$$

and our claim follows. Summing up, in case hypothesis i-) holds, we have $E_0(v) > E_0(v_0)$ for

$$0 < \|v - v_0\|_{L^1(N_\delta)} < \mu_0.$$

If i-) does not hold then one of the following cases would occur:

ii-) $\tilde{v}$ is not constant $\mathcal{H}^1$-a.e. along $\ell_{\tilde{t}}$, for a.e. $\tilde{t} \in (\delta/2, \delta)$.

iii-) $\tilde{v}$ is not constant $\mathcal{H}^1$-a.e. along $\ell_{-\tilde{t}}$, for a.e. $\tilde{t} \in (\delta/2, \delta)$.

iv-) $\tilde{v} \equiv \alpha\ \mathcal{H}^1$-a.e along $\ell_{\tilde{t}}$ and $\tilde{v} \equiv \beta$, $\mathcal{H}^1$-a. e. along $\ell_{-\tilde{t}}$, for a. e. $\tilde{t} \in (\frac{\delta}{2}, \delta)$.

Define $\Delta \subset (0, \delta)$ by

$$\Delta = \left\{ \tilde{t} \in (0,\delta) : \int_0^L \left\{ |\tilde{v} - \tilde{v}_0|(s,\tilde{t}) J_\Sigma(s,\tilde{t}) + |\tilde{v} - \tilde{v}_0|(s,-\tilde{t}) J_\Sigma(s,-\tilde{t}) \right\} ds > \frac{4\mu}{\delta} \right\}$$

with μ and δ given above. Therefore $|\Delta| < \delta/4$.

If iv-) holds then by choosing $\mu < \delta L$ we obtain:

$$\int_0^L \left\{ |\tilde{v} - \tilde{v}_0|(s,\tilde{t})J_\Sigma(s,\tilde{t}) + |\tilde{v} - \tilde{v}_0|(s,-\tilde{t})J_\Sigma(s,-\tilde{t}) \right\} ds = 2L(\beta - \alpha) > \frac{4\mu}{\delta} \, ,$$

that is, $\tilde{t} \in \Delta$. Hence for a.e. $\tilde{t} \in (\delta/2, \delta)\backslash\Delta$, either ii-) or iii-) holds. Futhermore if $k_m = \min_{X \in \overline{\Omega}} k(X)$ then with $\tilde{v}_j$ the approximant functions given above:

$$\int_0^L \left\{ \tilde{k}(s,\tilde{t}) \left| \frac{\partial \tilde{v}_j(s,\tilde{t})}{\partial s} \right| + \tilde{k}(s,-\tilde{t}) \left| \frac{\partial \tilde{v}_j(s,-\tilde{t})}{\partial s} \right| \right\} ds \geq$$

$$\geq k_m \left\{ \text{ess } V_0^L \left[\tilde{v}_j(\cdot,\tilde{t}) \right] + \text{ess } V_0^L \left[\tilde{v}_j(\cdot,-\tilde{t}) \right] \right\} \, .$$

For a.e. $\tilde{t} \in (\delta/2, \delta)\backslash\Delta$ it follows from the fact that either ii-) or iii-) holds that

$$\liminf_{j\to\infty} V_0^L \left[\tilde{v}_j(\cdot,\tilde{t}) \right] \geq \text{ess } V_0^L \left[\tilde{v}(\cdot,\tilde{t}) \right] \geq (\beta - \alpha)$$

with a similar inequality if $\tilde{t}$ is replaced by $-\tilde{t}$.

In order to obtain an estimate in $N_\delta \backslash N_{\delta/2}$, we integrate over $(\delta/2, \delta)\backslash\Delta$ and take limit as $j \to \infty$:

$$\lim_{j\to\infty} \left\{ \int_{\delta/2}^{\delta} \int_0^L \left[\tilde{k}(s,\tilde{t}) \left| \frac{\partial \tilde{v}_j(s,\tilde{t})}{\partial s} \right| + \tilde{k}(s,-\tilde{t}) \left| \frac{\partial \tilde{v}_j(s,-\tilde{t})}{\partial s} \right| \right] \right\} ds \, d\tilde{t} \geq$$

$$\geq k_m(\delta/4)(\beta - \alpha) \, .$$

Taking into account the definition of $\tilde{v}_0$ we have:

$$|\tilde{v}(s,\tilde{t}) - \tilde{v}(s,-\tilde{t})| \geq (\beta - \alpha) - \left\{ |\tilde{v}_0 - \tilde{v}|(s,\tilde{t}) + |\tilde{v}_0 - \tilde{v}|(s,-\tilde{t}) \right\} \, .$$

Since $|(0,\delta/2)\backslash\Delta| \geq \delta/4$ and $\tilde{v} \in BV(Q_\delta)$ it follows that there is $\bar{t} \in (0,\delta/2)\backslash\Delta$ such that $(s,\bar{t})$ and $(s,-\bar{t})$ are points of approximate continuity of $\tilde{v}(s,\tilde{t})$ for L^1-a.e. $s \in [0,L]$. Thus by using Lemmas 2.1, 2.2 and Fatou's Lemma we obtain the following estimate on $N_{\delta/2}$:

$$\begin{aligned}
\int_{N_{\delta/2}} k(X)d\|Dv\|(X) &= \lim_{k\to\infty} \int_0^L \int_{-\delta/2}^{\delta/2} \tilde{k}(s,t)\, \left|M_\Sigma^{-1}(s,t)\nabla_{s,t}\,\tilde{v}_j\right|\, J_\Sigma(s,t)\, dt\, ds \\
&\geq \lim\inf_{k\to\infty} \int_0^L \int_{-\delta/2}^{\delta/2} \tilde{k}(s,t) \left|\frac{\partial \tilde{v}_j}{\partial t}\right| J_\Sigma(s,t)\, dt\, ds \\
&\geq \nu_m \lim\inf_{k\to\infty} \int_0^L \left\{ \int_{-\delta/2}^{\delta/2} \left|\frac{\partial \tilde{v}_j}{\partial t}\right| dt \right\} ds \\
&\geq \nu_m \lim\inf_{k\to\infty} \int_0^L \operatorname{ess} V_{-\delta/2}^{\delta/2} \left(\tilde{v}_j(s,t)\right) ds \\
&\geq \nu_m \int_0^L \operatorname{ess} V_{-\delta/2}^{\delta/2} \left(\tilde{v}(s,t)\right) ds \\
&\geq \nu_m \int_0^L |\tilde{v}(s,\bar{t}) - \tilde{v}(s,-\bar{t})|\, ds \\
&\geq \nu_m \int_0^L \{(\beta-\alpha) - [\,|\tilde{v}-\tilde{v}_0|(s,\bar{t}) + |\tilde{v}-\tilde{v}_0|(s,-\bar{t})\,]\}\, ds \\
&\geq \left[(\beta-\alpha)\nu_m L - \frac{4\mu\,\nu_m}{\delta\, J_m(-\delta,\delta)}\right],
\end{aligned}$$

where

$$J_m(-\delta,\delta) = \min_{-\delta\leq t\leq\delta} \{\min_{0\leq s\leq L} J_\Sigma(s,t)\,,\ \min_{0\leq s\leq L} J_\Sigma(s,-t)\}\,.$$

Summing up the previous inequalities:

$$\begin{aligned}
E_0(v) &= \int_{N_\delta} k(X)d\|Dv\|(X) = \lim_{j\to\infty} \int_{N_\delta} k(X)|\nabla v_j|dX \\
&= \lim\inf_{j\to\infty} \int\!\!\int_{Q_\delta(s,t)} \tilde{k}(s,t)\, \left|M_\Sigma^{-1}(s,t)\nabla_{s,t}\,\tilde{v}_j(s,t)\right|\, J_\Sigma(s,t)\, ds\, dt \\
&= \lim\inf_{j\to\infty} \int\!\!\int_{Q_\delta(s,t)} \tilde{k}(s,t)\, J_\Sigma(s,t) \sqrt{\left(\frac{\partial \tilde{v}_j}{\partial s}\right)^2 J_\Sigma^{-2}(s,t) + \left(\frac{\partial \tilde{v}_j}{\partial t}\right)^2}\, ds\, dt \\
&\geq \lim\inf_{j\to\infty} \int_{\delta/2}^{\delta} \int_0^L \left\{ \tilde{k}(s,\tilde{t}) \left|\frac{\partial \tilde{v}_j(s,\tilde{t})}{\partial s}\right| + \tilde{k}(s,-\tilde{t}) \left|\frac{\partial \tilde{v}_j(s,-\tilde{t})}{\partial s}\right| \right\} ds\, d\tilde{t} \\
&+ \lim\inf_{j\to\infty} \int_0^L \int_{-\delta/2}^{\delta/2} \tilde{k}(s,t) \left|\frac{\partial \tilde{v}_j(s,\tilde{t})}{\partial t}\right| J_\Sigma(s,t)\, dt\, ds \\
&\geq \left[(\beta-\alpha)\nu_m L + k_m(\beta-\alpha)(\delta/4) - \frac{4\mu\,\nu_m}{\delta\, J_m(-\delta,\delta)}\right] > (\beta-\alpha)\nu_m L = E_0(v_0)\ ,
\end{aligned}$$

as long as we take

$$\mu < \min\left\{\frac{k_m(\beta-\alpha)\,\delta^2\,J_m(-\delta,\delta)}{16\,\nu_m}\,,\ \frac{(\beta-\alpha)}{2}\delta L\right\}.$$

The conclusion can now be established by extending v_0 to be constant on each connected component of $\Omega\backslash\gamma$ and observing that $\|Dv_0\|(\,\Omega\backslash\gamma\,) = 0$.

3. Existence and Stability

With the reaction term $f(v)$ in (P_1), satisfying (f_1) and (f_2), given in the Introduction, we define the following C^1-function

$$f_c(v) = \begin{cases} f'(\alpha)(v-\alpha), & \text{for} \quad -\infty \le v < \alpha \\ f(v), & \text{for} \quad \alpha \le v \le \beta \\ f'(\beta)(v-\beta), & \text{for} \quad \beta < v \le \infty \end{cases}$$

and the corresponding functional

$$E_{\varepsilon,c}(v) = \begin{cases} \displaystyle\int_\Omega \left\{\varepsilon\, k_1(X)\,|\nabla v|^2 + \varepsilon^{-1} k_2(X)\,[F(\alpha) - F_c(v)]\right\} dX\,, & \text{if} \quad v \in H^1(\Omega) \\ \infty\,, & \text{otherwise} \end{cases}$$

where $F_c(v) = \displaystyle\int_\theta^v f_c$. Note that by virtue of (f_0), $E_{\varepsilon,c}$ is well defined. See [**Fi2**], for instance. Clearly the potential function

$$G_c(X,v) = k_2(X)\,[F(\alpha) - F_c(v)]$$

satisfies $G_c(X,\alpha) = G_c(X,\beta) = 0\,,\ G_c^{'} \in C^2(\Omega\times\mathbb{R}\,,\,\mathbb{R}^+)$ i.e., $G_c(X,v) > 0$ for any $X \in \Omega$ and $v \in \mathbb{R}\,,\ v \ne \alpha$ and $v \ne \beta\,$.

This is crucial to our analysis which involves results about Γ-convergence and explain why we have to work with f_c instead of the original function f.

Throughout this paper we set for simplicity,

$$\mathcal{L}_\varepsilon(v) = \varepsilon^2\,\mathrm{div}\,[k_1(X)\,\nabla v]\ .$$

Any local minimiser v_ε of $E_{\varepsilon,c}$, $v_\varepsilon \in H^1(\Omega)$, is a weak solution of the following problem:

$$\begin{aligned} &\mathcal{L}_\varepsilon(v) + k_2(X)\,f_c\,(v) = 0\,, X \in \Omega \\ &\nabla v \cdot \widehat{n}_1(X) = 0\,, X \in \partial\Omega \end{aligned} \qquad (P_{2,c})$$

Applying the results of Section 2 to the above defined functionals $E_{\varepsilon,c}$ we conclude that

$$\Gamma\left(L^1(\Omega)^-\right) - \lim_{\varepsilon\to 0} E_{\varepsilon,c}(v) = E_0(v)\ ,$$

where

$$E_0(v) = \begin{cases} C_0 \displaystyle\int_\Omega k(X)\, d\|D\mathcal{X}_{\{v=\beta\}}\|\ , & \text{if} \quad v \in BV(\Omega),\ v(X) \in \{\alpha,\beta\} \\ \text{a.e. in } \ \Omega\,, & 0 < |\{X : v(X) = \beta\}| <|\,\Omega| \\ \infty\,, & \text{otherwise} \end{cases}$$

and $C_0 = \displaystyle\int_\alpha^\beta [F(\alpha) - F_c(t)]^{1/2}\, dt\,.$

Applying Theorem 2.3 to E_0 above we obtain an isolated L^1-local minimiser v_0 of E_0.

Lemma 3.1. *With the above notation it holds that:*

3.1.i) *There is a family $\{v_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$ of L^1-local minimisers of $E_{\varepsilon,c}$ such that $v_\varepsilon \in C^{2,\alpha}(\overline{\Omega})$, $0 < \alpha < 1$, v_ε is a solution of ($P_{2,c}$). Also $\|v_\varepsilon - v_0\|_{L^1(\Omega)} \to 0$, as $\varepsilon \to 0$.*

3.1.ii) *For any $X \in \overline{\Omega}$, $\alpha < v_\varepsilon(X) < \beta$.*

3.1.iii) *With Ω_i and Ω_o given in the Introdution, it holds that for any compact set $K \subset \Omega_i (K \subset \Omega_0)$, $v_\varepsilon \to \beta\, (v_\varepsilon \to \alpha)$, uniformly in K, as $\varepsilon \to 0$.*

Proof: 3.1.i) The existence of the minimisers v_ε, as described is a direct application of Theorem 2.1. As for condition (2.1.i) of that theorem it has been proved in [**FT**], [**S**] and [**M**]. By a standard argument, v_ε is a weak solution of $\mathcal{L}_\varepsilon(v) + k_2(X)\, f_c\,(v) = 0$ in $H^1(\Omega)$. A bootstrap argument yields $v_\varepsilon \in C^{2,\alpha}(\bar{\Omega})$, $0 < \alpha < 1$. Then as usual we conclude that v_ε is a solution of ($P_{2,c}$), for $0 < \varepsilon \le \varepsilon_0$.

3.1.ii) This is a direct consequence of the maximum principle. Indeed, let $v_M = \max_{X\in\overline{\Omega}} v_\varepsilon(X)$, $v_\varepsilon(X_M) = v_M$ and suppose that $v_M \ge \beta$. Hence $f_c(v_M) \le 0$. Note that since $\|v_\varepsilon - v_0\|_{L^1(\Omega)} \to 0$, as $\varepsilon \to 0$, if ε_0 is small enough then $\sup\limits_{x\in\Omega} f'_c(v_\varepsilon) = K > 0$ and $f_c(t) + Kt$ is monotone increasing in the image of $v_\varepsilon(X)$, $X \in \overline{\Omega}$. So $(\mathcal{L}_\varepsilon - k_2 K)\, v_\varepsilon = -k_2\,[f_c(v_\varepsilon) + K v_\varepsilon]$ and since $\mathcal{L}_\varepsilon v_M = 0 \leq -k_2(X) f_c(v_M)$ we obtain

$$(\mathcal{L}_\varepsilon - k_2 K)\, v_M \leq -k_2(X)\,[f_c(v_M) + K v_M]\ .$$

Subtracting the two inequalities,

$$[\mathcal{L}_\varepsilon - k_2(X) K]\ (\omega) \geq 0 \quad \text{on} \quad \Omega\,, \quad \text{where} \quad \omega = (v_\varepsilon - v_M) \leq 0\ .$$

Suppose that there is $P \in \overline{\Omega}$ such that $\omega(P) = 0$. Since clearly $\omega \not\equiv 0$, the maximum principle implies that $P \in \partial\Omega$ and there it holds that $\dfrac{\partial \omega(P)}{\partial n_1} > 0$. Since $\dfrac{\partial \omega(P)}{\partial n_1} = 0$ on $\partial\Omega$ we conclude that $\omega < 0$ on $\overline{\Omega}$, i.e., $v_\varepsilon(X) < v_M$ in $\overline{\Omega}$, which is a contradiction. Therefore $v_\varepsilon(X) < \beta$ in $\overline{\Omega}$. Similarly it can be proved that $v_\varepsilon(X) > \alpha$ in $\overline{\Omega}$.

3.1.iii) This is a consequence of a result of Caffarelli and Cordoba [**CC**] and (3.1.ii), for it can be checked that the potential function $G_c(X, v) = k_2(X)\,[F(\alpha) - F_c(v)]$, by virtue of hypotheses (f_1) and (f_2), satisfy the requirements in [**CC**]. These, for the reader's convenience, are presented below, where for simplicity in notation we write G instead of G_c and take $\theta = 0$, $\alpha = -1$, $\beta = -1$. The requirements are:

- $0 \leq G \leq 1$.
- $G(X, -1) = G(X, 1) = 0, \ \ \forall\, X \in \Omega$.
- $\inf\limits_{\substack{|t|<\lambda \\ X\in\Omega}} G(X, t) \geq \gamma(\lambda)$, where $\gamma(\lambda)$ is a decreasing, strictly positive function in the interval $[0, 1)$.
- There exists δ, $0 \leq \delta \leq 2$ and λ, $0 < \lambda < 1$, such that:

a-) $G(X, v) \geq C(1 - |v|)^\delta$, if $1 > |v| > \lambda$.

b-) In the case $0 < \delta \leq 2$, $G_v(X, v)$ is continuous on $\Omega \times (-1, 1)$ and satisfies the estimate

$$G_v(X, -1 + t) \geq C\, t^{\delta - 1}$$
$$G_v(X, 1 - t) \geq -C\, t^{\delta - 1}$$

if $t < \lambda$.

c-) In the case $\delta = 2$, $G_v(X, t)$ is increasing for t near -1 (respect. decreasing near 1).

Under these conditions, the uniform convergence of the level sets of the minimisers v_ε to the limiting curve γ, is proved. Hence (3.1.iii) follows. □

For future reference we define:

$$\begin{aligned} &\mathcal{L}_\varepsilon(v) + k_2(X)f(v) = 0 \ , X \in \Omega \\ &\nabla v \cdot \widehat{n}_1(X) = 0 \ , X \in \partial\Omega \end{aligned} \tag{P$_2$}$$

and also

$$E_\varepsilon(v) = \begin{cases} \displaystyle\int_\Omega \left\{ \frac{\varepsilon}{2} k_1(X) |\nabla v|^2 + \varepsilon^{-1} k_2(X) \left(F(\alpha) - F(v)\right) \right\} dX \, , & \text{if } \ v \in H^1(\Omega) \\ \infty \, , & \text{otherwise} \end{cases}$$

It is clear from Lemma 3.1 and the definition of f_c that the distinguished solution v_ε is a weak solution of (P_2)and hence a critical point of E_ε. It turns out however that v_ε is actually a local minimiser of E_ε above. The following theorem follows ideas set forth in [**Fi1**] about the use of variational techniques for some truncated functions.

Theorem 3.1. *With the above notation it holds that the local minimiser v_ε of $E_{\varepsilon,c}$ is also a local minimiser of E_ε.*

Proof. For simplicity in the computations we rescale the functional. Define $\widetilde{E}_\varepsilon(v) = \varepsilon E_\varepsilon(v)$ and note that v_ε is a minimiser of $\widetilde{E}_\varepsilon$ if and only if v_ε is a minimiser of E_ε. For simplicity in notation we drop the "tilde" and therefore keep the same notation.

By hypothesis there is $\mu_0 > 0$ such that $E_{\varepsilon,c}(v) \geq E_{\varepsilon,c}(v_\varepsilon)$ for all $v \in H^1(\Omega)$ such that $|||v - v_\varepsilon|||_{H^1(\Omega)} \leq \mu_0$, where for convenience we shall use the norm

$$|||v|||^2_{H^1(\Omega)} = \frac{1}{2} \int_\Omega (\varepsilon^2 k_1(X) \, |\nabla v|^2 + v^2) dX \, ,$$

which is equivalent to the usual one in $H^1(\Omega)$. Note that by (3.1.i) and (3.1.ii) of Lemma 3.1, each v_ε, $0 < \varepsilon \leq \varepsilon_0$, is not only a $C^{2,\alpha}$-solution of ($P_{2,c}$) but of (P_2) as well. Define

$$D_{\mu_0}(v_\varepsilon) = \left\{ v \in H^1(\Omega) : |||v - v_\varepsilon|||_{H^1(\Omega)} \leq \mu_0 \right\} .$$

Then it suffices to show that $E_\varepsilon(v_\varepsilon) \leq E_\varepsilon(v)$ for all $v \in D_{\mu_0}(v_\varepsilon)$.

If this were not the case then since E_ε is weakly lower semicontinuous and D_{μ_0} is weakly compact in the $H^1(\Omega)$ topology we would infer the existence of $\tilde{v}_\varepsilon \in D_\mu$, for some μ, $0 < \mu \leq \mu_0$ such that

$$E_\varepsilon(\tilde{v}_\varepsilon) = \inf\{E_\varepsilon(v),\ v \in D_\mu(v_\varepsilon)\} < E_\varepsilon(v_\varepsilon).$$

We define $\Lambda : H^1(\Omega) \to \mathbb{R}$ by $\Lambda(v) = \left(|||v - v_\varepsilon|||^2_{H^1(\Omega)} - \mu\right)$. Then $\tilde{v}_\varepsilon$ minimizes E_ε on

$$\mathcal{M} = \{v \in D_\mu(v_\varepsilon) : \Lambda(v) = 0\}.$$

Morever $\mathcal{M}$ is a C^1-submanifold of codimension 1 and since E_ε is C^1 it follows that there is a Lagrange multiplier $\lambda_\mu \leq 0$ (see [**Fi2**]) such that

$$\langle E'_\varepsilon(\tilde{v}_\varepsilon), \zeta\rangle_{H^*,H^1} = \lambda_\mu \langle \Lambda'(\tilde{v}_\varepsilon), \zeta\rangle,\ \forall \zeta \in H^1(\Omega)$$

where H^* stands for the dual of $H^1(\Omega)$.

Keeping in mind that v_ε satisfies (P_2) we obtain:

$$\int_\Omega \left[\varepsilon^2(\lambda_\mu - 1)k_1(X)\nabla\tilde{v}_\varepsilon.\nabla\zeta \ + \ (\lambda_\mu\tilde{v}_\varepsilon \ + \ k_2(X)f(\tilde{v}_\varepsilon))\zeta\right]\, dX =$$

$$= \lambda_\mu \int_\Omega \left[-\varepsilon^2\nabla\cdot(k_1\nabla v_\varepsilon) \ + \ v_\varepsilon\right]\zeta\, dX,\ \forall\zeta \in H^1.$$

This means that $\tilde{v}_\varepsilon$ satisfies in the H^1-sense:

$$-\varepsilon^2(\lambda_\mu - 1)\nabla\cdot(k_1\nabla\tilde{v}_\varepsilon) \ + \ \lambda_\mu\tilde{v}_\varepsilon = -k_2(X)f(\tilde{v}_\varepsilon) \ + \ \lambda_\mu[k_2(X)f(v_\varepsilon) \ + \ v_\varepsilon].$$

By virtue of (f_0), $f(\tilde{v}_\varepsilon)$ and $f(v_\varepsilon)$ are in $L^2(\Omega)$ and since $\lambda_\mu \leq 0$ (essencial here) a standard regularity argument yields $\tilde{v}_\varepsilon \in H^2(\Omega)$ and as such the directional derivative $\dfrac{\partial\tilde{v}_\varepsilon}{\partial\hat{n}_1}$ is defined a.e. on $\partial\Omega$ since $\partial\Omega$ is C^3.

Integrating by parts,

$$\varepsilon^2(\lambda_\mu - 1)\int_{\partial\Omega}\zeta k_1\nabla\tilde{v}_\varepsilon\cdot\hat{n}_1 ds + \int_\Omega\left[-\varepsilon^2(\lambda_\mu - 1)\nabla\cdot(k_1\nabla\tilde{v}_\varepsilon) + \lambda_\mu\tilde{v}_\varepsilon + k_2 f(\tilde{v}_\varepsilon)\right]\zeta dX =$$

$$= \lambda_\mu\int_\Omega\left[-\varepsilon^2\nabla\cdot(k_1\nabla v_\varepsilon) + v_\varepsilon\right]\zeta\, dX,\ \ \forall\zeta \in H^1(\Omega).$$

As usual now we take $\zeta \in H^1_0(\Omega)$ to conclude that $\tilde{v}_\varepsilon$ satisfies

$$-\varepsilon^2(\lambda_\mu - 1)\nabla\cdot(k_1\nabla\tilde{v}_\varepsilon) \ + \ k_2 f(\tilde{v}_\varepsilon) = -\varepsilon^2\lambda_\mu\nabla\cdot(k_1\nabla v_\varepsilon) \ + \ \lambda_\mu(v_\varepsilon - \tilde{v}_\varepsilon)$$

and substitute back to obtain $\nabla \tilde{v}_\varepsilon \cdot \hat{n}_1 = 0$ on $\partial\Omega$.

Since v_ε satisfies (P_2), after some computation we further obtain:

$$\varepsilon^2(\lambda_\mu - 1)\nabla \cdot [k_1 \nabla(\tilde{v}_\varepsilon - v_\varepsilon)] - \lambda_\mu(\tilde{v}_\varepsilon - v_\varepsilon) = k_2(X)[f(\tilde{v}_\varepsilon) - f(v_\varepsilon)]$$
$$\nabla(\tilde{v}_\varepsilon - v_\varepsilon) \cdot \hat{n}_1 = 0 \text{ on } \partial\Omega$$

By virtue of (f_0) the Nemitski operator $v \longrightarrow f(v)$ from L^2 to L^2 is continuous and then we evoke Schauder estimates to conclude that $\| \tilde{v}_\varepsilon - v_\varepsilon \|_{H^2(\Omega)}$ can be made as small as desired by making μ smaller, if necessary. But $\Omega \subset \mathbb{R}^2$ and then the Sobolev inclusion $W^{2,2}(\Omega) \hookrightarrow C(\bar{\Omega})$ allows us to conclude that $\alpha < \tilde{v}_\varepsilon(X) < \beta$, $\forall X \in \Omega$, if μ is taken sufficiently small.

This implies that $E_{\varepsilon,c}(\tilde{v}_\varepsilon) = E_\varepsilon(\tilde{v}_\varepsilon)$.

Finally from our hypothesis and the fact that $\alpha < v_\varepsilon < \beta$, we obtain

$$E_{\varepsilon,c}(\tilde{v}_\varepsilon) < E_{\varepsilon,c}(v_\varepsilon),$$

where $|||\tilde{v}_\varepsilon - v_\varepsilon|||_{H^1(\Omega)} \leq \mu_0$, which is against our hypothesis. □

Lemma 3.2. *Let $\{v_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ be the family of minimisers of E_ε, provided by Theorem 3.1. Then:*

3.2.i) v_ε *is a classical solution of (P_2).*

3.2.ii) *If $\{\lambda_n\}_{n=1,2,\ldots}$ is the sequence of eigenvalues of the problem:*

$$\begin{aligned} &\mathcal{L}_\varepsilon(\psi) + k_2(X)f'(v_\varepsilon)\psi = \lambda\psi \;, X \in \Omega \\ &\nabla\psi \cdot \hat{n}_1(X) = 0 \;, X \in \partial\Omega \;. \end{aligned} \qquad \text{(LP)}$$

Then $\lambda_n \leq 0$, $n = 1, 2, \ldots$ and λ_1 is a simple eigenvalue of the operator $\mathcal{L}_\varepsilon + k_2(X)f'(v_\varepsilon)$.

Proof: 3.2.i) It follows from the fact that v_ε is a L^1-local minimiser of E_ε that v_ε is a weak solution of (P_2)and thus a classical solution since $v_\varepsilon \in C^{2,\alpha}(\overline{\Omega})$, $0 < \alpha < 1$.

3.2.ii) It follows from the fact that

$$< E''_\varepsilon(v_\varepsilon)\psi \,,\, \psi > \; \geq 0 \,, \quad \text{for all} \quad \psi \in H^1(\Omega)$$

and from the variational characterization of the eigenvalues of (LP) that $\lambda_n \leq 0$, $n = 1, 2, \ldots$. As for the second statement it is a well known result that follows from an application of the Krein-Rutman theorem.

□

We should not expect to have $\lambda_1 [\mathcal{L}_\varepsilon + k_2(X) f'(v_\varepsilon)] < 0$ unless some additional symmetry hypothesis either on the function f or on the domain Ω are required.

However even in the case in which $\lambda_1 = 0$, a fairly complete picture of the stability of v_ε can be given, as the proof of the next lemma shows.

Lemma 3.3. *Let $\{v_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ be the family of stationary solution of (P_1)provided by Theorem 3.1. Then*

3.3.i) *If $\lambda_1 < 0$, v_ε is strongly stable in $H^1(\Omega)$.*

3.3.ii) *If $\lambda_1 = 0$, v_ε is stable in $H^1(\Omega)$.*

Proof. While (3.3.i) is well known , (3.3.ii) is just an application of Theorem 6.2.1. in [**H**] along with the existence of a Lyapunov function. Although in [**H**] the proof of this theorem is rendered for sectorial operators in the fractional power space it also holds for our case. Actually (P_1)written in an abstract form, defines a smooth dynamical system in $H^1(\Omega)$.

Specifically if $\lambda_1 = 0$ then by Lemma 3.2 it is a simple eigenvalue. Thus there is a local one-dimensional critical manifold $W_c(v_\varepsilon)$, tangent to $\{\psi_1\}$, ψ_1 : principal eigenfunction corresponding to $\lambda_1 = 0$ of (LP), such that if v_ε is stable in $W_c(v_\varepsilon)$ then it is also stable in $H^1(\Omega)$.

As for the (local) stability of v_ε in $W_\varepsilon(v_\varepsilon)$, it suffices to note that if $v(t, X)$ is a solution of (P_1)with $v(t_0, X) = v_0 \in W_c(v_\varepsilon)$ then

$$\frac{d}{dt} E_\varepsilon(v(t,X)) \leq -\int_\Omega v_t^2(t,X)dX\,, \quad \text{for} \quad t \geq t_0\,. \qquad \square$$

Summing up, all of these results establish the proof of Theorem 1.1.

Remark 3.1. Due to the local nature, in the spatial variable, of the argument used in the proof of Theorem 2.3, our result can be easily used to treat some cases in which γ is a simple but not closed curve. For instance, let $\gamma(s)$, $0 \leq s \leq L$, be a simple curve satisfying $\gamma(0) = P_1 \in \partial\Omega$, $\gamma(L) = P_2 \in \partial\Omega$, with $\partial\Omega$ locally concave at P_1 and P_2 and

$\dot{\gamma}(0)$ $(\dot{\gamma}(L))$ normal to $\partial\Omega$ at $P_1(P_2)$. In particular $\partial\Omega$ can be a segment of line in a neighbourhood of P_1 and\or P_2.

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